Dynamic Moral Hazard with Limited Commitment

Andong Yan

UC Riverside

Research Question

- Would a principal and an agent sign a contract while both of them understand the contract would not be (fully) implemented?
- What contracts are feasible when we explicitly allow for contract default and renegotiation/termination?
- How does the principal provide incentive (or not provide at all) in this limited commitment environment?

Model Setup

- agent makes full commitment, principal makes limited commitment
- common discount factor and discount rate $r = \frac{1}{1+\rho}$
- outcome $y_t \in \{0, 1\}$
- payoff to principal y if $y_t = 1$; 0 otherwise
- agent action $a_t \in \{0, 1\}$
- outcome distribution
  - $\mathbb{P}(y_t = 1 | a_t = 1) = \alpha$; $\mathbb{P}(y_t = 0 | a_t = 1) = 1 - \alpha$
  - $\mathbb{P}(y_t = 1 | a_t = 0) = \beta$; $\mathbb{P}(y_t = 1 | a_t = 0) = 1 - \beta$
  - $0 < \beta < \alpha < 1$
- payoff to agent
  - wage $w_t \geq 0$
  - shirk rent $\gamma > 0$ if $a_t = 0$; 0 otherwise
- infinite periods $t = 1, 2, 3, \ldots$

Model Setup

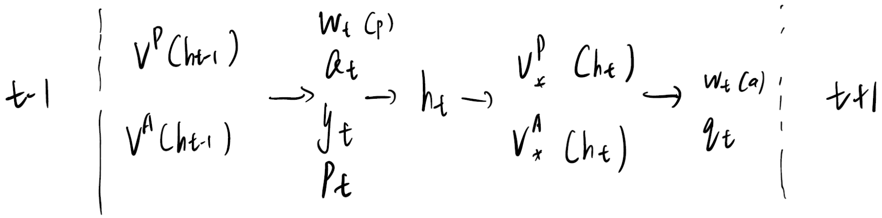
- Exogenous Contract State
  - $p_t \in \{0, 1\}$
  - $\mathbb{P}(p_t = 1) = \kappa$; $\mathbb{P}(p_t = 0) = 1 - \kappa$
  - $p_t = 1$, contract could be terminated by the principal
- Contract Termination
  - principal decision $q_t \in \{0, 1\}$ (Regardless of p_t)
  - principal's termination cost $c \geq 0$ if $q_t = 1$
  - agent's termination compensation $R \geq 0$ if $q_t = 1$
- Contract History $h_t = \{y_1, ..., y_t\}$
- Stopping Time $\tau = \min\{t \mid p_t = 1 \wedge q_t = 1\}$
- Principal Strategy

$$\sigma = \sigma(h_t) = \{w(h_t), q(h_t)\}$$

- Agent Strategy

$$a = a(h_{t-1})$$

Timeline



Expected Payoffs

Expected Payoffs at the beginning of period t, given history h_{t-1}

Principal Payoff $V^P(h_{t-1}, \sigma, a)$

$$= \mathbb{E}[\sum_{t \leq s \leq \tau} e^{-\rho(s-t)}(y_s - w_s) + e^{-\rho(\tau-t)}(-c + V^P)|h_{t-1}] \quad \text{given } \tau \text{ is bounded}$$

$$= \mathbb{E}[\sum_{s=t}^{\infty} e^{-\rho(s-t)}(y_s - w_s)|h_{t-1}] \quad \text{given } \tau \text{ is infinite}$$

Agent Payoff $V^A(h_{t-1}, \sigma, a)$

$$= \mathbb{E}[\sum_{t \leq s \leq \tau} e^{-\rho(s-t)}(w_s + \gamma(1 - a_s)) + e^{-\rho(\tau-t)}R|h_{t-1}] \quad \text{given } \tau \text{ is bounded}$$

$$= \mathbb{E}[\sum_{s=t}^{\infty} e^{-\rho(s-t)}(w_s + \gamma(1 - a_s))|h_{t-1}] \quad \text{given } \tau \text{ is infinite}$$

Optimal Contract

- We denote $V^P(\sigma, a)$ and $V^A(\sigma, a)$ the payoffs at the beginning of period 1
- If a contract (σ, a) satisfies

$$V^A(h_t, \sigma, a) \geq V^A(h_t, \sigma, \bar{a})$$

for all $\bar{a} = \{\bar{a}(h_t)\}$ and for all h_t, then it is **incentive compatible**.

- The optimal contract is (σ, a) such that σ maximizes $V^P(\sigma, a)$ and is incentive compatible.

Optimal Contract

- From now on, for simplicity, we suppress the arguments σ, a in the payoff functions
- At period 1, let $a_1 = a(h_0)$, we define $V_*^P(h_1), V_*^A(h_1)$ as following:

$$
\begin{aligned}
V_*^P(h_1) &= p_1[q(h_1)(-c + rV^P) + (1 - q(h_1)(-w(h_1) + rV^P(h_1))] \\
&\quad + (1 - p_1)[-w(h_1) + rV^P(h_1)] \\
V_*^A(h_1) &= p_1[q(h_1)R + (1 - q(h_1)(w(h_1) + rV^A(h_1))] \\
&\quad + (1 - p_1)[w(h_1) + rV^A(h_1)]
\end{aligned}
$$

- i.e., $V_*^P(\cdot), V_*^A(\cdot)$ are the expected payoffs after the realization of y_t, p_t but before the principal's decision q_t and payment transfer w_t
- Note that we also suppress the argument p_t in $V_*^P(\cdot), V_*^A(\cdot)$

Optimal Contract

- We can rewrite V^A, V^P as

$$V^P = \mathbb{E}[y_1 + V_*^P(h_1)|h_0]$$

$$V^A = \mathbb{E}[(1 - a_1)\gamma + V_*^A(h_1)|h_0]$$

- Next, we will reduce the dynamic problem to a static variational problem (Spear and Srivastava 1987)
- Briefly, we will define functions U^P, U^A to update both parties' expected payoffs for period $t+1$ conditional on the period t outcomes and their initial expected payoffs V^P, V^A.

Reduction of the Problem

- Let $\mathcal{V}^{\mathcal{P}} = \{V^P(h_t)\}$ and $\mathcal{V}^{\mathcal{A}} = \{V^A(h_t)\}$ where theses values are calculated at the optimal contract
- For each v in $\mathcal{V}^{\mathcal{A}}$, consider solving the optimization problem:
- maximize V^P at period 1 subject to agent receiving v and incentive compatible constraint

Reduction of the Problem

- Then, we denote $U^P(v)$ the principal's payoff in the solution to this problem
- Also, we have $a(v)$ the agent's action, $\sigma(v, y)$ the principal decisions, and $U^A(v, y)$ be the agent's payoff at period $t = 2$
- By construction, if $v = V^A(h_{t-1})$, we should have

$$U^P(v) = V^P(h_{t-1}),\ U^A(v, y) = V^A(h_t),\ a(v) = a(h_{t-1}),\ \sigma(v, y) = \sigma(h_t)$$

Optimal Contract

We could characterize the optimization problem by four functions:

$$
\begin{aligned}
U^A &: \mathcal{V}^{\mathcal{A}} \times \mathbb{R} \rightarrow \mathcal{V}^{\mathcal{A}} \\
U^P &: \mathcal{V}^{\mathcal{A}} \rightarrow \mathcal{V}^{\mathcal{P}} \\
\sigma &: \mathcal{V}^{\mathcal{A}} \times \mathbb{R} \rightarrow \mathbb{R} \times \{0,1\} \\
a &: \mathcal{V}^{\mathcal{A}} \rightarrow \{0,1\}
\end{aligned}
$$

which satisfies the following (necessary) conditions:

Necessary Conditions

1. $v = \mathbb{E}[(1-a(v))\gamma + V_*^A(v,y)] \geq \mathbb{E}[(1-a)\gamma + V_*^A(v,y)]$ for all $a \in \{0,1\}$
2. $U^A(v,y) = \mathbb{E}[1-a(U^A(v,y))\gamma + V_*^A(U^A(v,y),y')]$ for all $v \in \mathcal{V}^{\mathcal{A}}$,$y \in \{0,1\}$
3. $U^P(v) = \mathbb{E}[y + V_*^P(v,y)]$ for all $v \in \mathcal{V}^{\mathcal{A}}, y \in \{0,1\}$

where

a. $V_*^P(v,y) = p_t[q(v,y)(-c+rV^P) + (1-q(v,y))(-w(v,y)+rU^P(U^A(v,y))]$
$+(1-p_t)[-w(v,y)+rU^P(U^A(v,y))]$ for all $v \in \mathcal{V}^{\mathcal{A}}, y \in \{0,1\}$

b. $V_*^A(v,y) = p_t[q(v,y)R + (1-q(v,y))(w(v,y)+rU^A(v,y)]$
$+(1-p_t)[w(v,y)+rU^A(v,y)]$ for all $v \in \mathcal{V}^{\mathcal{A}}, y \in \{0,1\}$

Claims

We could make following claims :

- $w(v, y)$ is weakly increasing in y
- $w(v, 0) = 0$ for all v
- $U^A(v, y)$ is weakly increasing in y
- $U^P(v)$ is weakly decreasing and concave (without proof yet)

This implies the principal's payoff should not be dominated by the convex combination of any other two potential allocations.

Result

- Contract has no termination: $q(v, y) = 0$ for all v, y
  $\Rightarrow$ Stopping time $\tau = \infty$
  $\Rightarrow$ Infinite period contract
- Contract has termination: $\exists v, y$ such that $q(v, y) = 1$
  $\Rightarrow$ Random stopping time τ which is bounded above
  $\Rightarrow$ Finite period contract

Result

Proposition

An infinite static contract with no termination is feasible (but maybe suboptimal) for the following two scenarios:
(1) $a(v) = 0$, $w(v, y) = 0$, $q(v, y) = 0$ *for all* v, y; *(only possible choice if* $c = 0$*)*
(2) $a(v) = 1$, $w(v, 1) = w^* = \frac{\gamma}{\alpha}$, $w(v, 0) = 0$, $q(v, y) = 0$ *for all* v, y.

In these two scenarios, $V^P(\cdot)$, $V^A(\cdot)$ assign constant values for all h_t.

In (1), incentive compatibility is automatically satisfied. In (2), incentive compatibility could be satisfied by providing $w \geq \frac{\gamma}{\alpha}$.

Moreover, in (2), to ensure $q(\cdot) = 0$, the principal's participation constraint needs to satisfy $c > w(v, 1) = \frac{\gamma}{\alpha}$.

(1) yields principal $V^P = \frac{\beta y}{\rho}$ and (2) yields $V^P = \frac{\alpha(y - w^*)}{\rho}$.

We will assume (2) is feasible and strictly better than (1).

Result

Proposition

Assume $c < \frac{\gamma}{\alpha}$ and $y \geq \frac{\gamma}{\alpha - \beta}$, in the optimal contract, the principal will induce at least finite periods of $a = 1$.
The optimal dynamic contract payoff to the principal has a lower bound equal to payoff of the static contract which induces $a = 1$ for all periods.

We will keep this assumption in the following sections.

Note that since $U^P(v)$ is weakly decreasing and $U^P(v)$ is bounded below by contract (2), v is also bounded above by what contract (2) yields, which is $v \leq V^A = \frac{\gamma}{\rho}$ (equal to contract (1)).

Proposition

The optimal contract payoff to the agent has a upper bound equal to payoff of the static contract which requires $a = 1$ or $a = 0$ for all periods.

Result

Proposition

For any dynamic contract with no termination, there exists v^* *such that for* $v \geq v^*$,

$$a(v) = 0, w(v, y) = 0, U^A(v, y) = \frac{v - \gamma}{r}, U^P(v) = \beta y + rU^P(\frac{v - \gamma}{r})$$

for all y, *i.e., the contract requires* $a = 0$ *and deducts credits from the agent.*

Result

Proposition

(1) q(v,0)=1 $\Rightarrow$ *q(v,1)=1;*
(2) q(v,1)=0 $\Rightarrow$ *q(v,0)=0*

Proposition

q(v,y) is weakly increasing in y
q(v,y) is weakly increasing in v for $v \in \{v|w(v,y) = 0\}$

Proposition

For contract starts with $q(v,y) = 0$, *for* $v \in \{v|w(v,y) = 0\}$, $\exists \underline{v} < \bar{v}$ *such that*

(1) for $v < \underline{v}, q(v,y) = 0$
(2) for $v \in [\underline{v}, \bar{v}], q(v,0) = 0, q(v,1) = 1$
(3) for $v > \bar{v}, q(v,y) = 1$ *(if applies)*

for all y